

Phase transition of social learning collectives and “Echo chamber”

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Abstract

We study a simple model for social learning agents in a restless multi-armed bandit. There are N agents, and the bandit has M good arms that change to bad with the probability q_c/N . If the agents do not know a good arm, they look for it by a random search (with the success probability q_I) or copy the information of other agents’ good arms (with the success probability q_O) with probabilities $1 - p$ or p , respectively. The distribution of the agents in M good arms obeys the Yule distribution with the power-law exponent $1 + \gamma$ in the limit $N, M \rightarrow \infty$ and $\gamma = 1 + \frac{(1-p)q_I}{pq_O}$. The system shows a phase transition at $p_c = \frac{q_I}{q_I + q_O}$. For $p < p_c$ ($> p_c$), the variance of N_1 per agent is finite (diverges as $\propto N^{2-\gamma}$ with N). There is a threshold value N_s for the system size that scales as $\ln N_s \propto 1/(\gamma - 1)$. The expected value of the number of the agents with a good arm N_1 increases with p for $N > N_s$. For $p > p_c$ and $N < N_s$, all agents tend to share only one good arm. If the shared arm changes to be bad, it takes long time for the agents to find another good one. $E(N_1)$ decreases to zero as $p \rightarrow 1$, which is referred to as the “Echo chamber.”

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I. INTRODUCTION

Recently, social physics has become an active research field, and many studies have been devoted to understanding the social phenomena and interacting human behaviors [1–6]. In the process of these studies, social learning plays the key role as a fundamental interaction among humans [1, 2]. Here, social learning is a learning process through observing and imitating others' behaviors [7, 8]. In economics, the expected return or expected utility is maximized in the modelling of human behaviors. However, in many cases, such estimation is difficult or impossible for humans because of the limited capability of calculation and information gathering. Rationality is bounded and humans cannot avoid the uncertainty in decision making [9, 10]. In such uncertain situations, it is rational to observe others and copy their choices. Social learning is easy, as the cost is lower than individual learning, where one obtains information with one's effort [7, 11]. Individual learning has the advantage that the obtained information is more accurate and newer than the information obtained by social learning. Even so, the cost of the former learning is generally very high, and this cannot be compensated for by the accuracy of the obtained information [12].

As social learning is a process of copying information among agents and agents change their behaviors and make decisions based on that information, this process plays the role of interaction among agents. As the interaction is strong, the statistical mechanical approach is promising [13, 14]. One example is the phase transition caused by the information cascade [15, 16]. Information cascade is the tendency to choose the majority choice even if one thinks the minority option is correct or optimal [17, 18]. In laboratory experiments, at most 63 subjects answered two-choice questions one by one after observing the previous subjects' choices [19, 20]. The sequential choice processes were described as non-linear Pólya urn process. It was observed that as the difficulty of the question increases, the number of stable states changes. The change in the number of the stable states induces a nonequilibrium discontinuous phase transition in the infinite population limit [16]. The performance of humans is greatly improved by social learning, and this is called the collective intelligence effect [21, 22]. However, it can also cause suboptimal macroscopic behaviors where the majority choice is wrong.

Many kinds of social animals undertake social learning and make collective decisions [23, 24]. Owing to the abovementioned drawback of social learning, it is important to

investigate when, from whom, and how one copies others. These questions have been studied in behavioral ethology [7, 8, 25]. In 2010, a paper reported a tournament of social learning agent programs in a restless multi-armed bandit (rMAB). A multi-armed bandit is analogous to the “one-armed bandit” slot machine but with multiple “arms,” each with a distinct payoff [12]. We call an arm with a high payoff a good arm. The term “restless” means that the payoffs change randomly. Agents maximize their payoffs by exploiting an arm, searching for a good arm at random (individual learning), and copying an arm exploited by other agents (social learning). In the tournament, strategies that relied heavily on social learning were found to be remarkably successful. However, when all agents always adopt social learning in searching an arm, the performance was very bad [12]. They do not bring in new information, and outdated information is prevalent among them. This kind of suboptimal state is called an “Echo chamber,” in which wrong and outdated information spreads among agents, and the performance is severely degraded [1]. By using the data in the online trade market e-Toro, it was reported that the excessive observation of others’ portfolios leads to the deterioration of the performance [1].

In this paper, we study a simple model for social learning agents in an rMAB. Previous studies focused on the optimal social learning strategy in a nonuniform and changing environment [26, 27]. Here, we focus on the collective behavior of social learning agents in the context of statistical physics. The proposed model is deeply related with other models. If one regards an arm as a color and an agent as a ball, it is a kind of generalized Pólya urn process [28]. The bad arm’s color is black, and the good arms’ colors are different from other arms’ colors. If an agent knows an arm, the color of the ball is the color of the arm. If a randomly chosen ball’s color is black, it is replaced with a ball with another color. Contrary to the generalized Pólya urn process, in an rMAB, all balls of a color change to bad randomly. In social learning, the probability that a good arm is chosen is proportional to the number of agents who know it. It works as a preferential attachment mechanism of evolving networks [29, 30]. The model is also a variant of the cultural neutral model [31]. In this context, each arm corresponds to an option, a city, a product, a song, a fashion, an idea, etc. In the model, in each turn, a new agent is added, and this agent will copy the choices of previous agents. Instead of the spontaneous change in the arms in an rMAB, a new option is added, and the agent will choose this new option with a certain probability. The cultural neutral model explains right-skew socioeconomic distributions and the continuous turnover

observed empirically in the distributions [32, 33].

We organize the paper as follows. We introduce a simple model for social learning collectives in an rMAB in Sec. II. Sec. III is devoted to the analysis for the solvable case where the rMAB has one good arm. In Sec. IV, we study the model with multiple good arms using the mean field approximation. The distribution of agents among good arms obeys the Yule distribution in the infinite population and good arm limit. We also estimate a critical system size N_s , beyond which the system is in equilibrium and the performance improves with the rate of social learning. In Sec. V, we show that the system shows a phase transition. We describe how the performance behaves in each phase and explain the nature of the “Echo chamber.” Sec. VI is devoted to the conclusions and to discussing future problems.

II. MODEL

There are N agents, and the rMAB has M good arms. We label the agents by $n \in \{1, 2, \dots, N\}$ and the good arms by $m \in \{1, 2, \dots, M\}$. We identify all bad arms with the arm $m = 0$. In each turn, one agent is randomly chosen, and he exploits his arm if he knows a good one. If he does not know a good arm, he looks for it by a random search (individual learning) or copies the information of other agents’ good arms (social learning) with probabilities $1 - p$ or p , respectively. In the random search, we denote the probability that the agent successfully knows a good arm as q_I . As there are M good arms, each arm is chosen with the probability q_I/M . In the copy process, he can know a good arm with the success probability q_O if there is at least one agent who knows a good one. If there are multiple agents who know a good arm, one of them is chosen randomly. The probability that arm m is chosen is proportional to the number of the agents who know it. After that, good arms change to bad with the probability q_C/N independently of each other. If a good arm m changes to bad, the agents who know it are forced to forget it and to know the bad arm $m = 0$. We define one Monte Carlo step (MCS) or one round as N turns. The probability that an arm does not change during one MCS (round) is $(1 - q_C/N)^N \simeq e^{-q_C}$.

In the agent tournament [12], agents copy the arm that is exploited in the previous round. In each round, all N agents perform their actions. In this case, we can estimate q_O as $q_O \simeq (1 - q_C/N)^{2N} \simeq e^{-2q_C}$. The reasoning for this estimation is that the copied agents exploited the arm N turns before. The information of the exploited arm was assumed to be

updated N turns before the exploit. Since the distribution of the payoff obeys an exponential distribution, q_I is small. In this paper, we assume $q_I < q_O$.

We denote the number of agents who know arm m after t as $K_m(t)$. As we are interested in the performance of the agents' system, we also introduce $N_1(t) \equiv \sum_{m=1}^M K_m(t)$, which is the number of agents who know a good arm after t . The number of agents who do not know a good arm is denoted as $N_0(t)$, and $N_0(t) = N - N_1(t) = K_0(t)$ holds.

In turn t , agent n is picked randomly among N agents. If this agent knows a good arm, the agent exploits it to obtain a payoff. Otherwise, the agent copies with the probability p and searches randomly with the probability $1 - p$. If $N_1 = 0$, the agent cannot know a good arm by copying. In terms of $\vec{K} = (K_0, \dots, K_M)$, the probability for the change $\Delta K_m = 1$ by an agent action with the condition $\vec{K} = \vec{k}$ is written as

$$p_m(\vec{k}) \equiv \Pr(\Delta K_m = 1) = \frac{N_0}{N} \cdot \left(A \cdot (1 - \delta_{N_1,0}) \cdot \frac{k_m}{N_1} + B \cdot \frac{1}{M} \right). \quad (1)$$

Here, A and B are defined as $A \equiv pq_O$, $B \equiv (1 - p)q_I$. We write the probability that there is no change in \vec{K} by an agent action as $p_{NC}(\vec{k}) \equiv 1 - \sum_{m=1}^M p_m(\vec{k})$. We summarize the change in \vec{K} by the transfer matrix T^A as

$$T^A(\vec{k}'|\vec{k}) \equiv \Pr(\vec{K} = \vec{k} \rightarrow \vec{k}') = \left(p_{NC}(\vec{k}) \prod_{m=1}^M \delta_{k_m, k'_m} + \sum_{m=1}^M p_m(\vec{k}) \delta_{k'_m, k_m+1} \prod_{l \neq m} \delta_{k'_l, k_l} \right).$$

After an agent action, all good arms change to be bad with the probability q_c/N independently. If arm m changes to bad, K_m becomes zero. We write the change in \vec{K} using the transfer matrix T^C as

$$T^C(\vec{k}'|\vec{k}) \equiv \Pr(\vec{K} = \vec{k} \rightarrow \vec{k}') = \prod_{m=1}^M \left(\left(1 - \frac{q_C}{N}\right) \cdot \delta_{k'_m, k_m} + \frac{q_C}{N} \cdot \delta_{k'_m, 0} \right).$$

We write the probability function for $\vec{K}(t) = \vec{k}$ as $P(\vec{k}, t)$.

$$P(\vec{k}|t) \equiv \Pr(\vec{K}(t) = \vec{k}).$$

The Chapman–Kolmogorov equation for $P(\vec{k}|t)$ is

$$P(\vec{k}|t+1) = \sum_{\vec{k}'} \sum_{\vec{k}''} T^C(\vec{k}|\vec{k}') T^A(\vec{k}'|\vec{k}'') P(\vec{k}''|t). \quad (2)$$

Initially, no agent knows good arms, and $\vec{K}(0) = (N, 0, \dots, 0)$. The initial conditions for $P(\vec{k}|0)$ are $P((N, 0, \dots, 0)|0) = 1$ and $P(\vec{k}|0) = 0$ for $\vec{k} \neq (N, 0, \dots, 0)$. If $p = 1$ or $q_I = 0$, no agent can know a good arm forever. We assume that $p < 1$ and $q_I > 0$ to avoid such trivial results.

If N is sufficiently large, the equilibrium between $N_1(t)$ and $N_0(t)$ should be realized in the stationary state. We denote their equilibrium values as $N_1 \equiv \lim_{t \rightarrow \infty} N_1(t)$ and $N_0 \equiv \lim_{t \rightarrow \infty} N_0(t)$, respectively. The equilibrium condition between N_0 and N_1 is given as

$$\frac{N_0}{N} \cdot (A + B) = N_1 \cdot \frac{q_C}{N}. \quad (3)$$

The left-hand side shows the expected number of agents who come to know a good arm, which is given by the product of the probability that a randomly chosen agent knows a bad arm, N_0/N , and the probability that the agent succeeds in knowing a good one, $A + B$. If N is large, the number of agents who know a good arm is always positive, and the probability for successful copying is q_O . The right-hand side shows the expected number of agents whose arm changes from good to bad. We solve the condition with the constraint $N_0 + N_1 = N$, and we have

$$\frac{N_1}{N} = \frac{A + B}{q_C + A + B} = \frac{pq_O + (1 - p)q_I}{q_C + pq_O + (1 - p)q_I}. \quad (4)$$

As we assume $q_I < q_O$, N_1/N becomes an increasing function of p . In the limit $p \rightarrow 1$, N_1/N converges to $q_O/(q_O + q_C)$. As will be shown later, if p is larger than a threshold value p_c , the system size N required for maintaining the equilibrium becomes extremely large. If N is smaller than the required value, $N_1(t)$ shows an oscillatory behavior, as the system is trapped in the state $N_0 = N$ for a long time. The expected number of N_1/N decreases with p .

III. EXACT SOLVABLE $M = 1$ CASE

When there is only one good arm, the state of the system is described by $K_1 = N_1 = n \in \{1, 2, \dots, N\}$. In this case, we can obtain the stationary probability function $P(n) = \lim_{t \rightarrow \infty} \Pr(N_1(t) = n)$ and estimate several quantities. Here, we explain some important results. The details of the derivations are given in Appendix A.

The expectation value of N_1/N is evaluated as

$$\begin{aligned} E(N_1/N) &= \frac{1 + a/N}{a + 1} \cdot (1 - P(0)), \\ P(0) &= \frac{q_C}{q_C + (N - q_C)B}. \end{aligned} \quad (5)$$

In the limit $N \rightarrow \infty$, $P(0) \rightarrow 0$, and we obtain

$$\lim_{N \rightarrow \infty} E(N_1/N) = \frac{1}{a + 1} = \frac{A + B}{q_C + A + B}.$$

The result is in accordance with the result from Eq. (4).

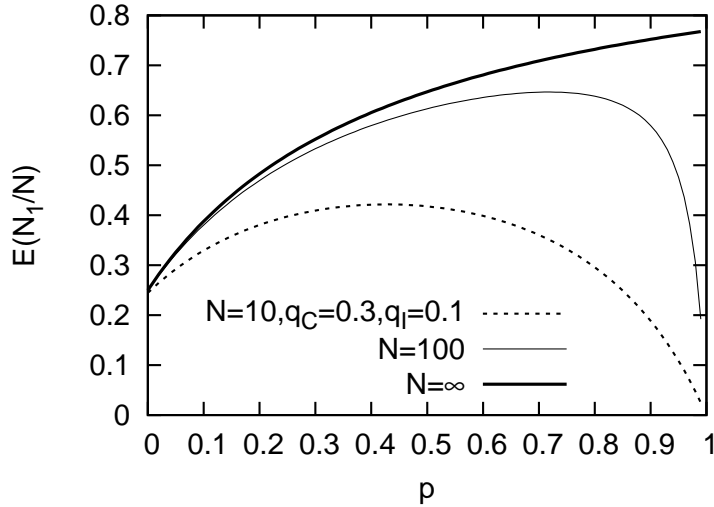


FIG. 1. Plots of $E(N_1)/N$ vs. p for $M = 1$. We plot Eq. (5) for the plot $N < \infty$ and Eq. (4) for the plot $N = \infty$. $q_C = 0.3, q_I = 0.1, q_O = 1$, and $N \in \{10, 10^2, 10^4, \infty\}$.

Figure 1 shows a plot of $E(N_1/N)$ vs. p . As can be clearly seen, as N increases, the curves for $E(N_1/N)$ converge to the curves for the results in the limit $N \rightarrow \infty$. $E(N_1/N)$ for finite N increases with p up to a point and then decreases with p . As N increases, the peak point for $E(N_1/N)$ moves rightward and converges to 1 in the limit $N \rightarrow \infty$.

To understand the mechanism for the bad performance for large p , we derive the self-consistent equation for n_1 , which is the limiting value of the expectation value of N_1 , $n_1(t) \equiv E(N_1(t))$.

$$n_1 = \lim_{t \rightarrow \infty} n_1(t) = \lim_{t \rightarrow \infty} \sum_n P(n|t)n.$$

It is given as

$$\frac{n_1}{N} = \left(1 - \frac{q_C}{N}\right) \cdot \frac{A + B - AP(0)}{q_C + (1 - \frac{q_C}{N})(A + B)}. \quad (6)$$

This is another expression for the result in Eq.(5). In the limit $N \rightarrow \infty$, $P(0) = 0$, and $q_C/N = 0$. We get the same result in Eq. (4). The numerator in Eq. (6) decreases as $P(0)$ increases. As $p \rightarrow 1$ and $B = (1 - p)q_I \rightarrow 0$, $P(0)$ increases to 1, and the numerator reduces to 0. As there is only one good arm in the system, agents are forced to share the arm. If the arm changes to be bad, there is no agent who knows a good arm, and copying fails. If p is large and B is small, agents cannot know a good arm for a long duration. Thus, $E(N_1/N)$ becomes small compared with the equilibrium value, where many good arms are known by agents and copying always succeeds as $N_1 > 0$.

IV. MEAN FIELD ANALYSIS FOR $N, M \rightarrow \infty$

In this section, we study the distribution of N agents in M good arms in the limit $N, M \rightarrow \infty$. We denote the number of good arms with k agents at t as $M_k(t)$. As there are N agents, $k \in \{0, \dots, N\}$. In terms of $\vec{K}(t)$, $M_k(t)$ is written as $M_k(t) = \sum_{m=1}^M \delta_{K_m(t), k} \cdot M_k(t)$ satisfies $\sum_{k=0}^N M_k(t) = M$. In terms of $M_k(t)$, we can write $N_1(t)$ as $N_1(t) = \sum_{k=1}^N M_k(t) \cdot k$.

The probabilistic rules for M_k by an agent action are summarized as

$$\begin{aligned}
\Pr(\Delta M_0 = -1) &= (1-x)B \cdot \frac{M_0}{M}, \\
\Pr(\Delta M_0 = 0) &= 1 - \Pr(\Delta M_0 = -1), \\
\Pr(\Delta M_k = 1) &= (1-x) \left(A \cdot \frac{(k-1)M_{k-1}}{N_1} + B \cdot \frac{M_{k-1}}{M} \right), \\
\Pr(\Delta M_k = -1) &= (1-x) \left(A \cdot \frac{kM_k}{N_1} + B \cdot \frac{M_k}{M} \right), \\
\Pr(\Delta M_k = 0) &= 1 - (\Pr(\Delta M_k = 1) + \Pr(\Delta M_k = -1)), \\
k &\in \{1, 2, \dots, N-1\} \\
\Pr(\Delta M_N = +1) &= (1-x) \left(A \cdot \frac{(N-1)M_{N-1}}{N_1} + B \cdot \frac{M_{N-1}}{M} \right), \\
\Pr(\Delta M_N = 0) &= 1 - \Pr(\Delta M_N = +1), \\
x &\equiv N_1/N.
\end{aligned}$$

In the rules, the terms with the coefficient A correspond to successful copying. They should be considered to be zero if $N_1 = 0$. $N_1 = \sum_k M_k \cdot k$ obeys the next probabilistic rule.

$$\begin{aligned}
\Pr(\Delta N_1 = 1) &= (1-x) \left[A \cdot \left(\frac{N_1 - N \cdot M_N}{N_1} \right) + B \cdot \left(1 - \frac{M_N}{M} \right) \right], \\
\Pr(\Delta N_1 = 0) &= 1 - \Pr(\Delta N_1 = 1).
\end{aligned} \tag{7}$$

By changing arms, M_k changes with the binomial probabilities $B(M_k, p_c/N)$:

$$\Pr(\Delta M_k = -\Delta m_k) = \binom{M_k}{\Delta m_k} \left(\frac{q_C}{N} \right)^{\Delta m_k} \left(1 - \frac{q_C}{N} \right)^{M_k - \Delta m_k}.$$

M_0 increases by $\sum_{k=1}^N \Delta m_k$, and N_1 decreases by $\sum_{k=1}^N \Delta m_k \cdot k$.

We denote the expectation values of $M_k(t)$ and $N_1(t)$ as $m_k(t)$ and $n_1(t)$, respectively. The recursive relation for $m_0(t)$ is

$$m_0(t+1) = m_0(t) - \left(1 - \frac{n_1}{N} \right) \cdot \left(1 - \frac{q_C}{N} \right) \cdot B \cdot \frac{m_0(t)}{M} + \frac{q_C}{N} \cdot (M - m_0(t)).$$

The second term on the right-hand side of the equation corresponds to a decrease by random search with the probability $(1 - N_1/N) \cdot B \cdot M_0/M$. With the probability q_C/N , the arm changes to bad, and the net decrease is reduced by $(1 - q_C/N)$. The third term corresponds to the change of good arms. There are $M - m_0$ good arms with nonzero agents, and they

change to bad with the probability q_C/N .

The recursive relations for $m_k, k \geq 1$ are

$$m_k(t+1) = \left(1 - \frac{q_C}{N}\right) \left\{ m_k(t) + \left(1 - \frac{n_1(t)}{N}\right) \left(A \cdot \frac{(k-1)m_{k-1}(t) - km_k(t)}{n_1(t)} + B \cdot \frac{m_{k-1}(t) - m_k(t)}{M} \right) \right\}, \quad k \in \{1, \dots, N-1\},$$

$$m_N(t+1) = \left(1 - \frac{q_C}{N}\right) \left\{ m_N(t) + \left(1 - \frac{n_1(t)}{N}\right) \left(A \cdot \frac{(N-1)m_{N-1}(t)}{n_1(t)} + B \cdot \frac{m_{N-1}(t)}{M} \right) \right\}.$$

The prefactor $(1 - q_C/N)$ on the right-hand side corresponds to the changes of arms. In the bracket $\{, \}$, the terms proportional to $(1 - n_1/N)$ correspond to a change by copying and a random search. As $q_I > 0$ and $p < 1$, we can assume that $n_1(t) > 0$ for $t > 0$. At $t = 0$, $n_1(0) = 0$, and the term proportional to A should be zero. Using the above relations, we write the recursive relation for $n_1 = \sum_k m_k k$ as

$$n_1(t+1) = \left(1 - \frac{q_C}{N}\right) \cdot \left\{ n_1(t) + \left(1 - \frac{n_1(t)}{N}\right) \left(A \cdot \frac{n_1(t) - Nm_N(t)}{n_1(t)} + B \cdot \frac{M - m_N(t)}{M} \right) \right\}. \quad (8)$$

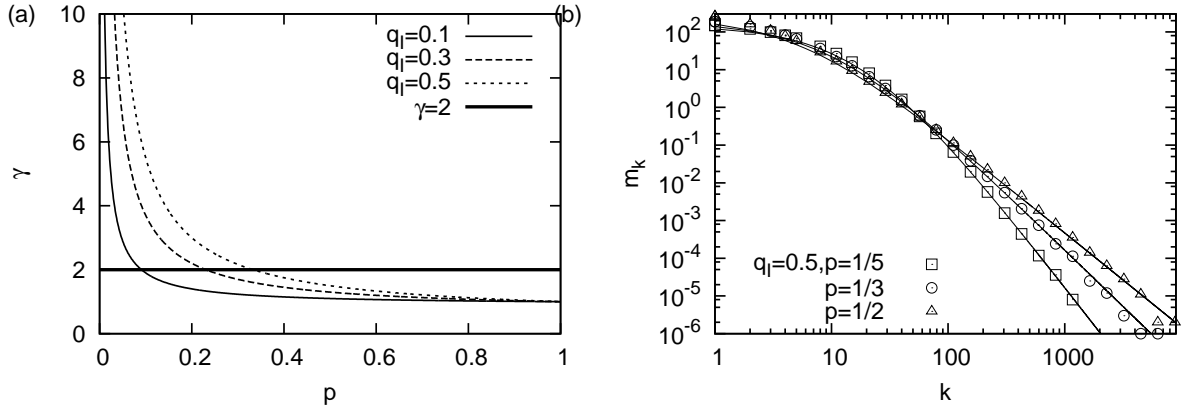


FIG. 2. (a) Plots of p vs. γ in Eq. (10). $q_C = 0.1, q_O = 1$, and $q_I \in \{0.1, 0.3, 0.5\}$. (b) Plots of m_k vs. k of the numerical (symbols) and theoretical results (solid curves) from Eq. (9). $q_O = 1.0$ and $q_I = 0.5$. $p = 1/5, 1/3, 1/2$ and $\gamma = 3, 2, 3/2$, respectively.

Here, we assume that the system is stationary for sufficiently large N . The stationary value of $n_1(t)/N$, which we denote x , is estimated by Eq. (4) as

$$x = \lim_{t \rightarrow \infty} n_1(t)/N = \frac{A + B}{A + B + p_c}.$$

$m_k(t)$ evolves to their stationary values, and we write them as $m_k \equiv \lim_{t \rightarrow \infty} m_k(t)$. We solve the above recursive relations by replacing $m_k(t+1)$ and $m_k(t)$ with m_k . m_0 is obtained as

$$m_0 = \frac{M}{1 + \alpha/\gamma}, \quad \alpha \equiv \frac{B \cdot Nx}{A \cdot M}, \quad \gamma \equiv \frac{xq_C}{(1-x)(1-q_C/N)A}.$$

m_k and m_N obey the following recursive relations:

$$m_k = \frac{(k-1) + \alpha}{k + \alpha + \gamma} \cdot m_{k-1}, \quad m_N = \frac{(N-1) + \alpha}{\gamma} \cdot m_{N-1}.$$

We obtain m_k and m_N as

$$\begin{aligned} m_k &= \frac{B(k + \alpha, \gamma + 1)}{B(\alpha, \gamma + 1)} \cdot m_0, \\ m_N &= \frac{B(N + \alpha, \gamma)}{B(\alpha, \gamma + 1)} \cdot m_0 = \frac{N + \alpha + \gamma}{\gamma} \cdot \frac{B(N + \alpha, \gamma + 1)}{B(\alpha, \gamma + 1)} \cdot m_0. \end{aligned} \quad (9)$$

$m_k, k < N$ obeys the Yule distribution [34, 35] (see Figure 2).

We take the limit $N, M \rightarrow \infty$ with $M/N = \beta$. γ is given as

$$\gamma = \frac{x}{1-x} \cdot \frac{q_C}{A} = \frac{A+B}{A} = 1 + \frac{(1-p)q_I}{pq_O}. \quad (10)$$

α is also written as

$$\alpha = \frac{Bx}{A\beta}.$$

We see that the power-law exponent for m_k is given by $1 + \gamma$.

$$m_k \propto k^{-1-\gamma} = k^{-(2+\frac{(1-p)q_I}{pq_O})}.$$

γ decreases with p , as in Figure 2. As γ is larger than one for $p < 1$ and $m_N \propto (N+\alpha)^{-\gamma}$, $m_N \rightarrow 0, Nm_N \rightarrow 0$ in the limit. The recursive relation for n_1 in Eq. (8) then becomes an equilibrium condition for n_1 when one replaces $n_1(t+1)$ and $n_1(t)$ by n_1 as

$$\frac{n_1}{N} = \left(1 - \frac{p_c}{N}\right) \cdot \left\{ \frac{n_1}{N} + \left(1 - \frac{n_1}{N}\right) \cdot \frac{A+B}{N} \right\}.$$

It reduces to the same equilibrium equation for $x = n_1/N$ in Eq. (3) in the limit $N \rightarrow \infty$.

The result is consistent with the assumption that n_1 converges to the stationary value.

To estimate the sufficient system size N_s for the stationarity of the system, we decompose the summation \sum_k for $x = n_1/N = \sum_{k=0} m_k k/N$ in the former part for $k \leq N_s$ and the latter part for $k > N_s$ as

$$x = \sum_{k=0}^{\infty} m_k k/N = \sum_{k=0}^{N_s} m_k k/N + \sum_{k=N_s+1}^{\infty} m_k k/N.$$

The latter part is then given as

$$\sum_{k=N_s+1}^{\infty} m_k k/N = \frac{B(\alpha + N_s, \gamma + 1)}{B(\alpha, \gamma + 1)} \frac{\alpha + \gamma + N_s}{\alpha + \gamma} \frac{\alpha + N_s}{\gamma - 1} \beta \propto N_s^{1-\gamma}.$$

If we set $N_s = 0$, as $x = \sum_{k=N_s+1}^{\infty} m_k k/N$, we obtain the self-consistent equation for x , which is the same as the equilibrium condition for n_1/N in Eq. (3). For the former term $\sum_{k=1}^{N_s} m_k k/N$ almost coincides with x , the latter term should be almost zero. As $p \rightarrow 1$, $\gamma \rightarrow 1$, and the convergence of the latter term to zero becomes remarkably slow. We estimate the ratio of the latter term to x as

$$\frac{1}{x} \sum_{k=N_s+1}^{\infty} m_k k/N = \frac{B(\alpha + N_s, \gamma + 1)}{B(\alpha, \gamma + 1)} \cdot \frac{\alpha + \gamma + N_s}{\alpha + \gamma} \cdot \frac{\alpha + N_s}{\alpha} \simeq \frac{B(\alpha + N_s, \gamma + 1)}{B(\alpha, \gamma + 1)} N_s^2.$$

The ratio is proportional to $N_s^{1-\gamma}$, and it converges to zero in the limit $N_s \rightarrow \infty$. If the ratio is smaller than some small value δ , N_s provides the adequate system size up to the ratio δ . N_s is then estimated as

$$\frac{B(\alpha + N_s, \gamma + 1)}{B(\alpha, \gamma + 1)} N_s^2 = \delta.$$

Since the left-hand side behaves as $N_s^{1-\gamma}$, we have $\ln N_s \propto \ln \delta / (\gamma - 1)$.

We find that $x = n_1/N$ is given by the equilibrium value in Eq. (4), and the agent distribution is scale-free, i.e., $m_k \sim (k + \alpha)^{-(\gamma+1)}$ if $N > N_s$. As p increases from zero to one, x increases from $q_I/(q_I + q_C)$ to $q_O/(q_O + q_C)$. At the same time, γ decreases to one, and the adequate system size N_s diverges. If N becomes smaller than N_s , the system is not in equilibrium, and N_1 shows strong fluctuations.

V. PHASE TRANSITION AND ECHO CHAMBER

We estimate the fluctuation of N_1 around the stationary value n_1 for $N > N_s$. A good arm changes to bad with the probability q_C/N . If the changed arm is known by k agents, $\Delta N_1 = -k$, and $\Delta M_k = -1$. Here, we assume that N is sufficiently large, and we can neglect the change in M_k . We replace M_k with their expectation values m_k in Eq. (9). Then, we write the probabilistic rule for $N_1 \in [0, N]$ as

$$\begin{aligned}\Pr(\Delta N_1 = 1) &= \left(1 - \frac{N_1}{N}\right) (A + B), \\ \Pr(\Delta N_1 = -k) &= \frac{q_C}{N} \cdot m_k.\end{aligned}$$

The first equation corresponds to the action of a randomly chosen agent. If $N_1 = 0$, we assume $\Pr(\Delta N_1 = 1) = B$, as copying fails. The second equation corresponds to the random change of arms. Here, we assume that there occurs only one arm change in M_k arms with k agents. Since we assume N is sufficiently large, the assumption is reasonable.

The expectation value of ΔN_1 is estimated as

$$E(\Delta N_1) = \left(1 - \frac{N_1}{N}\right) (A + B) - \frac{p_c}{N} \sum_{k=1}^{N_1} m_k \cdot k.$$

As $N > N_s$, $\sum_{k=1}^N m_k k \simeq Nx$, and $E(\Delta N_1)$ vanishes for $N_1 < N$, which we denote as $N_{1,c}$. If $\gamma > 2$, m_k decays rapidly, and the contribution from the tail part in $\sum_{k=1}^N m_k k$ is negligible. In this case, $N_{1,c} \simeq Nx$, and N_1 fluctuates around Nx . However, if $\gamma < 2$, the contribution from the tail part in $\sum_{k=1}^N m_k k$ is not negligible. $N_{1,c}$ is larger than Nx .

We estimate the size of the fluctuation of N_1 around $N_{1,c}$. We denote the variance of N_1 as $V_{N_1}(N) \equiv \text{Var}(N_1)$, where we write the system size dependence explicitly. As there are N agents, $V_{N_1}(N)$ is estimated by N times the variance of ΔN_1 . The variance of N_1 per agent is then estimated as

$$V_{N_1}(N)/N \equiv E((\Delta N_1)^2) = \left(1 - \frac{N_{1,c}}{N}\right)(A + B) + \sum_{k=1}^{N_{1,c}} m_k k^2.$$

$m_k \sim k^{-1-\gamma}$, and the second term diverges as $N_{1,c}^{2-\gamma}$ for $\gamma \leq 2$. As $N_{1,c} \propto N$, we have the

scaling relation for $V_{N_1}(N)$ as

$$V_{N_1}(N)/N \propto \begin{cases} N^{2-\gamma} & \text{for } \gamma < 2, \\ \ln N & \text{for } \gamma = 2, \\ N^0 & \text{for } \gamma > 2. \end{cases} \quad (11)$$

If $\gamma > 2$, $V_{N_1}(N)/N$ does not depend on the system size N and is finite in the limit $N \rightarrow \infty$. If $\gamma < 2$, $V_{N_1}(N)/N$ diverges as $N^{2-\gamma}$. For $\gamma = 2$, we see logarithmic divergence in $V_{N_1}(N)$. The system shows a phase transition as γ passes between the two phases with a different asymptotic behavior for $V_{N_1}(N)/N$. We denote the threshold value for p as p_c , which is given by

$$p_c = \frac{q_I}{q_I + q_O}. \quad (12)$$

If $p > p_c$, $\gamma < 2$, and $V_{N_1}(N)/N \propto N^{2-\gamma}$. If $p < p_c$, $\gamma > 2$, and $V_{N_1}(N)/N \propto N^0$. The left figure in Figure 3 shows the phase boundary in the (p, q_I) plane. We adopt $q_O \in \{1, 0.5, 0.1\}$. In general, p_c is an increasing function of q_I and a decreasing function of q_O . If $q_I = 0.5$ and $q_O = 1$, p_c is estimated as $1/3$.

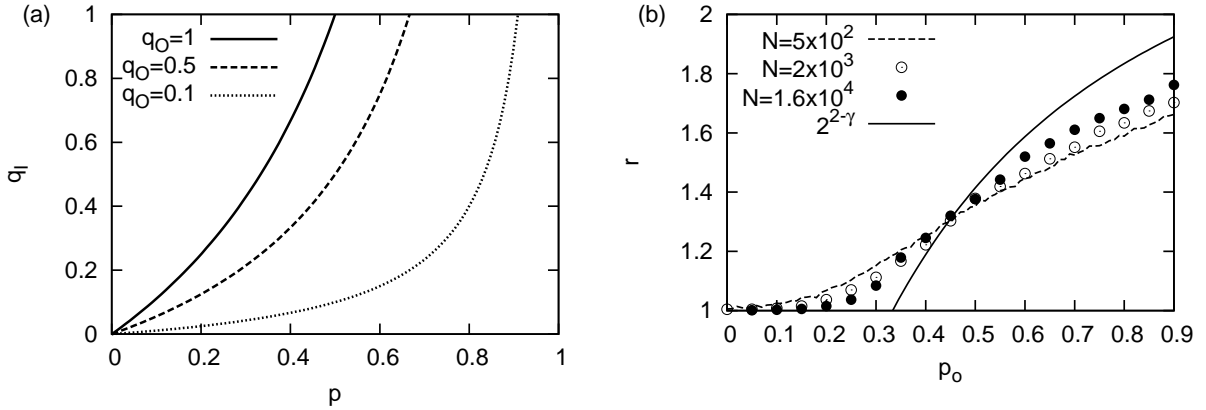


FIG. 3. (a) Phase diagram in the (p, q_I) plane. Plots of p_c vs. q_I in Eq. (12) for $q_O \in \{1, 0.5, 0.1\}$. (b) Plots of r vs. p . $q_C = 0.1, q_I = 0.5, q_O = 1, \beta = 0.1$, and $N \in \{500, 2000, 16000\}$. The thin solid line corresponds to the limit behavior $2^{2-\gamma}$ for $p > p_c = 1/3$.

To check the phase transition, we study the ratio of $V_{N_1}(N)/N$ for $N = N$ and $N = 2N$. We denote the ratio as r .

$$r \equiv \frac{V_{N_1}(2N)/2N}{V_{N_1}(N)/N}.$$

If $\gamma > 2$, $V_{N_1}(N)/N$ does not depend on N , and $r = 2^0 = 1$. If $\gamma < 2$, $V_{N_1}(N) \propto N^{2-\gamma}$, and $r = 2^{2-\gamma}$. Figure 3(b) plots r vs. p . We adopt $q_I = 0.5$, and $q_C = 0.1, q_O = 1$ and $\beta = 0.1$. We have performed Monte Carlo simulations with 10^6 MCS. The system sizes N are $N = 5 \times 10^2, 2 \times 10^3$, and 1.6×10^4 . If $p < p_c = 1/3$, r is almost one, and this suggests that $V_{N_1}(N)/N$ does not depend on N . If $p > p_c$, r deviates from one, which means that $V_{N_1}(N)/N$ depends on N . The numerical results for r almost coincide with $2^{2-\gamma}$ if p is not much larger than p_c and $N = 1.6 \times 10^4$. The scaling behavior for $V_{N_1}(N)$ holds for $N > N_s$. In order to check the scaling behaviors for $p \gg p_c$, it is necessary to study a system with larger size N .

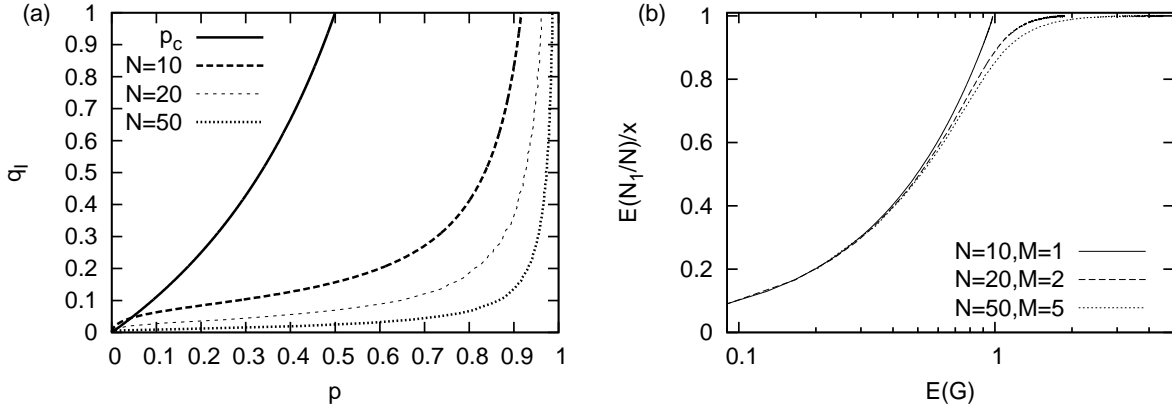


FIG. 4. (a) Plot of p_c and p_{EC} vs. q_I in the (p, q_I) plane. Plots of p_c vs. q_I in Eq. (12) for $q_O = 1$ in thick solid line. p_{EC} is defined as the threshold value where $E(N_1/N)/x = 0.9$. We adopt $q_c = 0.1, q_I = 0.5, \beta = 0.1$ and $N \in \{10, 20, 50\}$. (b) Plot of $E(N_1/N)/x$ vs. $E(G) \equiv E(\sum_m (1 - \delta_{K_m, 0}))$. We adopt the same parameters with the left figure.

At last, we study the ‘‘Echo chamber’’ state of the system. For large p , $E(N_1/N)$ becomes a decreasing function of p . We study how the performance of the agents is deteriorated for large p . We define the Echo chamber state where the ratio $E(N_1/N)/x$ is smaller than 0.9. We denote the threshold value as p_{EC} . We numerically estimate p_{EC} and show the results in Figure 4a. We set $p_c = 0.1, \beta = 0.1$, and $N \in \{10, 20, 50\}$. We see that the Echo chamber state is in the phase $p > p_c$, except for $N = 10, M = 1$, and $q_I \simeq 0$.

In Figure 4b, we plot $E(N_1/N)/x$ vs. the average number of good arms known by agents, which is denoted as $E(G)$. $E(G)$ is estimated as

$$E(G) \equiv E\left(\sum_m (1 - \delta_{K_m, 0})\right).$$

In the “Echo chamber” and $p > p_{EC} > p_c$, $E(N_1/N)/x$ is less than 0.9. In this case, as Figure 4b shows, the average number of known good arms is less than one. This suggests that only one good arm is shared by all agents on average in the “Echo chamber” state.

VI. CONCLUSIONS

We study a simple model for social learning agents in an rMAB. There are N agents and the bandit has M good arms that change to bad with the probability q_c/N . If the agents do not know a good arm, they look for it by a random search (with the success probability q_I) or copy the information of other agents’ good arms (with the success probability q_O) with the probabilities $1 - p$ or p , respectively. We summarize the results below.

1. $N \rightarrow \infty$ limit: The system is in equilibrium, and the number of agents who know a good arm N_1 is balanced with the number of the agents who do not N_0 . The ratio N_1/N becomes an increasing function of p for $p < 1$ and $q_I < q_O$.
2. $M = 1$ case: If N is finite, $E(N_1)/N$ decreases with p for large p . There is only one good arm, and agents cannot find a good one after the shared good arm changes to be bad if p is large.
3. $N, M \rightarrow \infty$ limit: The stationary distribution of agents in good arms $\{m_k\}$ is described by the Yule distribution $m_k \sim k^{-1-\gamma}$, and $\gamma = 1 + \frac{(1-p)q_I}{pq_O}$. N_1/N fluctuates around the equilibrium value in Eq. (4). The system shows a phase transition at $p_c = q_I/(q_I + q_O)$, and the asymptotic behavior of $\text{Var}(N_1)/N$ changes. If $p < p_c$, $\text{Var}(N_1)/N$ is finite in the limit $N \rightarrow \infty$. If $p > p_c$, $\text{Var}(N_1)/N \propto N^{2-\gamma}$ and diverges with N .
4. $N, M < \infty$ case: There is a threshold value N_s about the system size N , and N_s behaves as $\ln N_s \propto 1/(1 - \gamma)$. If $N > N_s$, the system is in equilibrium and multiple good arms are known by agents. $E(N_1)/N$ is almost the same as the equilibrium value. If $p < p_c$, N_s is small, and $N > N_s$ usually holds. If $p > p_c$, N_s becomes very large. If $N < N_s$, all agents tend to share one good arm, as in the case of $M = 1$, and $E(N_1)/N$ decreases with p , which is the “Echo chamber” state of social learning collectives.

At last, we point out several future problems. The first problem is the empirical verification of the predictions of the model. The model predicts that the social learning agents

show a phase transition, and the agents distribution has a scale-free nature. In general, the performance of agents improves with p , and the power-law exponent decreases with p . These predictions should be studied with the tournament results [12]. Experimental study is also an interesting research direction. Thus far, there have been some experimental studies of human collectives in an MAB [36] and rMAB[37, 38]. In these studies, the collective intelligence by social learning has been studied. This paper provides a new experimental design from the viewpoint of a phase transition. The second problem is the network effect. In this paper, we treat the case where the agents can copy any agent's arm. In reality, agents should have limited access to other agents, and the accessibility should be incorporated as a network[39]. The information of an arm is transmitted over the network, and there is a possibility of a phase with multiple arms, even for large p . As for the third problem, a multi-armed bandit with arms having different payoffs should be studied. In the setting of the agent tournament, the payoff obeys an exponential distribution, and an arm with high return is rare [12]. As it is natural that an agent exploits the arm or searches for a new arm depending on the return of his arm, the model should incorporate the exit of agents from good arms. In an evolving network model, incorporation of the fitness of nodes leads to the Bose condensation of edges [40]. The collective behaviors of agents in the rMAB with arms having different payoffs might be an interesting problem.

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Appendix A: Exact Solvable $M = 1$ case

The Chapman–Kolmogorov equation for $P(n|t) = \Pr(N_1(t) = n)$ is

$$\begin{aligned}
P(n|t+1) &= \sum_{n', n''} T^C(n|n') T^A(n'|n'') \cdot P(n''|t), \\
T^A(n|n') &= q(n') \delta_{n, n'} + p_1(n') \delta_{n-1, n'}, \\
T^C(n|n') &= \left(1 - \frac{q_C}{N}\right) \delta_{n, n'} + \frac{q_C}{N} \delta_{n, 0}, \\
p_1(n) &= \frac{N-n}{N} \cdot (A \cdot (1 - \delta_{n, 0}) + B), \\
q(n) &= 1 - p_1(n).
\end{aligned} \tag{A1}$$

The initial condition is $P(n|0) = \delta_{n, 0}$. $P(n|t)$ satisfies the following relation:

$$\begin{aligned}
P(n|t+1) &= \left(1 - \frac{q_C}{N}\right) \cdot q(n) P(n|t) + \left(1 - \frac{q_C}{N}\right) \cdot p_1(n-1) P(n-1|t) \quad \text{for } n \geq 1, \\
P(0|t+1) &= \frac{q_C}{N} + \left(1 - \frac{q_C}{N}\right) \cdot q(0) P(0|t).
\end{aligned}$$

We write the stationary probability function as $P(n) \equiv \lim_{t \rightarrow \infty} P(n|t)$. We solve $P(0)$ and $P(1)$ as

$$\begin{aligned}
P(0) &= \frac{q_C}{q_C + (N - q_C)B}, \\
P(1) &= \frac{B}{A + B} \cdot \frac{N}{a + (N - 1)} \cdot P(0) \\
a &= \frac{q_C}{(1 - q_C/N)(A + B)}.
\end{aligned} \tag{A2}$$

$P(n)$ satisfies the following recursive relation for $n \geq 2$:

$$P(n) = \frac{N - (n - 1)}{N - (n - a)} \cdot P(n - 1).$$

There is a threshold value for a . Depending on whether $a > 1$ or $a < 1$, the behavior of $p(n)$ changes drastically. We solve the recursive relation and obtain

$$P(n) = \begin{cases} \frac{B(N-n+a, 1-a)}{B(N-1+a, 1-a)} \cdot P(1) & \text{for } a < 1, \\ \frac{B(N, a-1)}{B(N-n+1, a-1)} \cdot P(1) & \text{for } a > 1, \\ P(1) & \text{for } a = 1. \end{cases} \quad (\text{A3})$$

As $B(a, b) \sim a^{-b}$,

$$P(n) \sim (N - n)^{a-1}.$$

We define the critical value p_c for p by the condition $a = 1$. We obtain

$$p_c = \frac{q_C / (1 - q_C / N) - q_I}{q_o - q_I}.$$

If $p < p_c$ and $a > 1$, $P(n)$ becomes a decreasing function of n . However, if $p > p_c$ and $a < 1$, $P(n)$ becomes an increasing function of n , and $P(n)$ has a fat tail.

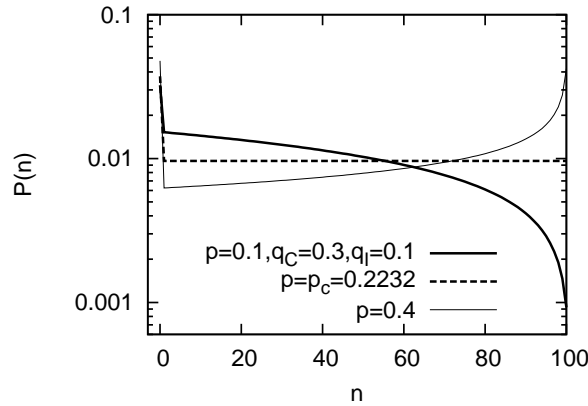


FIG. 5. Plots of $P(n)$ vs. n in Eq. (A3). $M = 1, N = 10^2, q_c = 0.3, q_I = 0.1, q_o = 1$, and $p \in \{0.1, p_c = 0.2232, 0.3\}$.

Figure 5 shows a plot of $P(n)$ for $N = 10^2, q_c = 0.3, q_I = 0.1$ and $q_o = 1$. In this case, $p_c \simeq 0.2233$. We choose $p \in \{0.1, p_c, 0.4\}$. For $p = 0.1$ and $a > 1$, $P(n)$ decreases with n . If

$p = p_c$ and $a = 1$, $P(n)$ becomes constant for $n \geq 1$. If $p > p_c$, $P(n)$ has a peak at $n = N$ in addition to the peak at $n = 0$.

The variance of N_1/N is evaluated as

$$\begin{aligned} V(N_1/N) &\equiv E((N_1/N)^2) - E(N_1/N)^2 \\ &= \frac{(2 + a/N)(1 + a/N)}{(a + 1)(a + 2)} \cdot (1 - P(0)) - E(N_1/N)^2. \end{aligned} \quad (\text{A4})$$

In the limit $N \rightarrow \infty$, we obtain

$$\lim_{N \rightarrow \infty} V(N_1/N) = \frac{a}{(a + 1)^2(a + 2)}.$$

As $\lim_{N \rightarrow \infty} V(N_1/N) > 0$, N_1/N has a wide distribution in the limit $N \rightarrow \infty$. In fact, N_1/N obeys a beta distribution $\text{beta}(1, a)$, and the correlation coefficient among M_n is $1/(a + 2)$ in this limit [41].

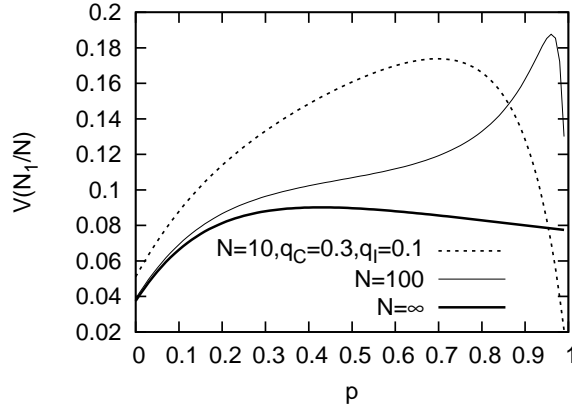


FIG. 6. Plots of $V(N_1/N)$ vs. p in Eq. (A4). $q_C = 0.3, q_I = 0.1, q_O = 1$, and $M = 1$. $N \in \{10, 10^2, 10^4, \infty\}$.

Figure 6 shows a plot of $V(N_1/N)$ vs. p . We adopt the same set for the parameters, as in Figure 5. As can be clearly seen, as N increases, the curve for $V(N_1/N)$ converges to that for the result in the limit of $N \rightarrow \infty$. We find no evidence of singular behavior in N_1 . This is an artifact of the case $M = 1$. The agents are forced to choose the unique good arm $m = 1$. With the change of the good arm, N_1 changes to zero. As the result, the limiting value of the variance of N_1/N becomes finite even for $p = 0$. If there are many good arms, agents can be distributed among multiple good arms. If $p = 0$, all agents search for a good arm

independently, which are distributed equally among all M arms. In this case, the variance of N_1/N is proportional to $1/M^2$. In the limit $p \rightarrow 1$, all agents share the same good arm, and the variance of N_1/N becomes finite in the limit $N \rightarrow \infty$, even if $M \rightarrow \infty$. This means that the variance of N_1 per agent is finite for small p and diverges with N for large p . This suggests that the system shows a phase transition in the limit $N, M \rightarrow \infty$.

We derive the self-consistent equation for n_1 , which is the limiting value of the expectation value of N_1 , $n_1(t) = E(N_1(t))$.

$$n_1 = \lim_{t \rightarrow \infty} E(N_1(t)) = \lim_{t \rightarrow \infty} \sum_n P(n|t)n$$

We write the recursive relation for $n_1(t)$ using the relations for $P(n|t)$ as

$$n_1(t+1) = \sum_n P(n|t+1) \cdot n = \left(1 - \frac{q_C}{N}\right) \left(1 - \frac{A+B}{N}\right) n_1(t) + \left(1 - \frac{q_C}{N}\right) (A+B - AP(0|t)).$$

We solve the self-consistent equation for $n_1 \equiv \lim_{t \rightarrow \infty} n_1(t)$ and obtain the expression in Eq. (6).